13/9/23 MATH2050A lecture

Announcements:

TA Information: Stephen Liu
sylui@math.cuhk.edu.hk.
LSB 222A
Office Hours by email appointment.

Goal:

- Discuss Order Properties of $\mathbb{R}$
- Introduce Completeness axiom for $\mathbb{R}$.

Recall

$$\mathbb{N} \subset \underset{\text{additive inverses}}{\mathbb{Z}} \subset \mathbb{Q}$$

$\mathbb{Q}$ – multiplicative inverses.

$\mathbb{N}$ – induction: $\mathbb{N}$ is s.t. if $m \in \mathbb{N}$, then $m+1 \in \mathbb{N}$.

Algebraic axioms for $\mathbb{Q}$. – make $\mathbb{Q}$ into a field. $\mathbb{R} \supset \mathbb{Q}$.

Motivating Observation: $\nexists\, q \in \mathbb{Q}$ s.t. $q^2 = 2$

Pf: Suppose on the contrary, that $\exists\, q \in \mathbb{Q}$ s.t. $q^2 = 2$.

We write $q = \frac{m}{n}$ for $m, n \in \mathbb{Z}$ and m, n coprime.

Then $q^2 = 2 \Rightarrow \left(\frac{m}{n}\right)^2 = 2 \Rightarrow m^2 = 2n^2$. $\Rightarrow m^2$ is even.

m^2 even $\Rightarrow m$ even. (sps. $r = 2s-1$ odd, then $r^2 = (2s-1)^2$
$= 4s^2 - 4s + 1$
$= 2(2s^2 - 2s) + 1$
odd.).

We can write $m = 2t$.

Then we get $(2t)^2 = 2n^2$

$$\overset{2}{\cancel{4}}t^2 = \cancel{2}n^2. \Rightarrow n^2 = 2t^2 \Rightarrow n \text{ is even.}$$

But this contradicts the fact that m, n are coprime

But first, ordering properties for $\mathbb{R}$:

<u>Axiom</u>: There is a non-empty subset $\mathbb{P} \subset \mathbb{R}$ s.t.

1) if $a \in \mathbb{R}$ then exactly one of the following holds:

$$\left.\begin{array}{l} \cdot\ a = 0 \\ \cdot\ a \in \mathbb{P} \\ \cdot\ -a \in \mathbb{P} \end{array}\right\} \text{Trichotomy Property}$$

$\mathbb{R} = \mathbb{P} \cup \{0\} \cup \{a \in \mathbb{R} \text{ s.t. } -a \in \mathbb{P}\}$.

disjoint union.

2) If $a, b \in \mathbb{P}$, then $a + b \in \mathbb{P}$.

3) If $a, b \in \mathbb{P}$, then $ab \in \mathbb{P}$.

<u>Def/Notation</u>: If $a \in \mathbb{P}$, then we write $a > 0$.

If $a \in \mathbb{P} \cup \{0\}$, then we write $a \geq 0$.

If $-a \in \mathbb{P}$, then we write $a < 0$.

If $-a \in \mathbb{P} \cup \{0\}$, then we write $a \leq 0$.

If $a-b \in \mathbb{P}$, then $a>b$. ...

Trichotomy Prop $\Rightarrow$ exactly one of the 3 holds: $a<b$, $a=b$, $a>b$.

Thm: a) If $a>b$, $b>c$, then $a>c$

b) If $a>b$, then $a+c>b+c$ for any $c \in \mathbb{R}$.

c) If $a>b$, $c>0$, then $ac>bc$

If $a>b$, $c<0$, then $ac<bc$

d) If $a \in \mathbb{R}$, $a \neq 0$, then $a^2>0$

e) $1>0$

f) if $n \in \mathbb{N}$, then $n>0$.

Pf: c) $a>b$ means $a-b \in \mathbb{P}$. If $c \in \mathbb{P}$, then $\underset{\uparrow \text{(3) above}}{c \cdot (a-b) \in \mathbb{P}}$

so $ca>cb$.

OTOH, if $c<0$, then $-c \in \mathbb{P}$. so $cb-ca \overset{\text{distributionaxiom}}{=} (-c)(a-b) \in \mathbb{P}$.

So $cb > ca$.

Pf of (a), (b), (d) - (f) left as exercise.

for (e), use (d).

(f), use (e) and induction.

Completeness Axiom for $\mathbb{R}$

Back to $\sqrt{2}$: $\sqrt{2} = 1.41421356237.....$

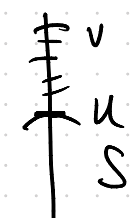
is there some number m s.t. $m^2 \approx \sqrt{2}$? notion of limit/approximation.

Def: Given $S \subseteq \mathbb{R}$, S is <u>bounded from above</u> if $\exists m \in \mathbb{R}$ s.t. $s \leq m$ $\forall s \in S$. (bdd. from above).

(l.u.b).

- (Least Upper Bound): Given $S \subseteq \mathbb{R}$ bdd from above, $u \in \mathbb{R}$ is a least upper bound of S if

1) $u \geq s$ $\forall s \in S$
2) if $\exists v \in \mathbb{R}$ s.t. $v \geq s$ $\forall s \in S$, then $u \leq v$.



Prop: l.u.b. of S is unique:

Pf: Suppose $u, \hat{u}$ are l.u.b. of S. Then by (2), $u \le \hat{u}$.

But also $\hat{u} \le u$ by interchanging the roles of $u, \hat{u}$.

So $u = \hat{u}$ (by trichotomy property).

Rmk: So it makes sense to talk about *the* l.u.b. of S, and we write $u = \sup S$. The *supremum* of S.

Def: Infimum of S (greatest lower bound of S). u is g.l.b. of S if

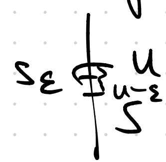
1) $u \le s \ \forall s \in S$

2) if $v \le s \ \forall s \in S$, then $v \le u$.

Alternative Def'n of l.u.b.: For $S \subseteq \mathbb{R}$ bdd. from above, u is $\sup S$ if

1) $u \geq s \ \forall s \in S$

2) $\forall \varepsilon > 0$, $\exists s_\varepsilon \in S$ s.t. $u - \varepsilon < s_\varepsilon$.



Exercise: Show that these two definitions are equivalent.

Note: if $u = \sup S$, it is not necessarily the case that $u \in S$.

Example: $(0,1) := \{x \in \mathbb{R} : 0 < x < 1\}$. $1 = \sup(0,1)$, but $1 \notin (0,1)$.

Axiom (Completeness of $\mathbb{R}$): "fills in the gaps in $\mathbb{Q}$"

If $S \subseteq \mathbb{R}$ is non-empty and bdd. from above, then $\exists u \in \mathbb{R}$ s.t. $u = \sup S$.

$\Rightarrow$ if $S \subseteq \mathbb{R}$ is non-empty and bdd. from below, then $\exists u \in \mathbb{R}$ s.t. $u = \inf S$.

Next Time: consequences of completeness.